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QUASI-MULTIPLIERS

BY

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ABSTRACT. A quasi-multiplier m on an algebra A is a bilinear mapping from $A \times A$ into itself such that $m(ax, yb) = am(x, y)b$ for all $a, x, y, b \in A$. An introduction to the theory of quasi-multipliers on Banach algebras with minimal approximate identities is given and applications to C^*-algebras and group algebras are developed.

1. **Introduction.** Ronald Larsen's excellent book *An Introduction to the Theory of Multipliers* treats the subject of multipliers on commutative Banach algebras. The extension of the concept to a noncommutative Banach algebra takes on several possible forms, each of which, however, suffers from some defect. There may be too few double multipliers for the purpose at hand, and neither left multipliers nor right multipliers are symmetric with respect to multiplication. All of these generalizations are special cases of a further generalization, the quasi-multiplier. The result of [1], which states that the quasi-multipliers of a C^*-algebra may be identified with precisely those elements of the bidual which are weakly continuous on the state space, presents at least some evidence that the quasi-multiplier may be the proper generalization.

The principal apparent defect of quasi-multipliers is that, at least a priori, there seems no way to multiply them together. This defect is partially repaired in the present paper for a considerable class of Banach algebras.

The first part of the paper is an introduction to the general theory and the second, an investigation of the quasi-multipliers of some particular Banach algebras.

2. **General theory.** For any ring R, a bilinear mapping $m|R \times R \to R$ such that

$$m(ab, cd) = am(b, c)d \quad \text{for all } a, b, c, d \in R \tag{1}$$

will be called a *quasi-multiplier*. For a topological ring R, we will write $QM(R)$ for the set of all separately continuous quasi-multipliers.

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Throughout this paper, $(A, \| \ \|)$ will be a Banach algebra with minimal approximate identity, i.e., a net $\{x_\alpha\}$ such that $\|x_\alpha\| \leqslant 1$ for each index α, and $\lim_\alpha \|ax_\alpha - a\| = \lim_\alpha \|x_\alpha a - a\| = 0$ for all $a \in A$.

THEOREM 1. *Suppose* $m|A \times A \to A$ *satisfies condition* (1) *above. Then* m *is a jointly continuous quasi-multiplier and, for all* $a, b, c \in A$,

$$(2) \qquad m(ab, c) = am(b, c) \quad \text{and} \quad m(a, bc) = m(a, b)c.$$

PROOF. Let $a, b, c \in A$ and $\alpha \in C$. The Hewitt-Cohen factorization theorem ([4, Theorem 32.23]) enables a choice of $w, x, y, z \in A$ such that $a = wx, b = wy$, and $c = wz$. Then

$$m(ab, c) = m(awy, wz) = awm(y, w)z = am(wy, wz) = am(b, c),$$

which proves the first half of (2); the second follows analogously. Further, by (2)

$$m(\alpha a, b) = m(\alpha wx, b) = \alpha wm(x, b) = \alpha m(wx, b) = \alpha m(a, b)$$

and

$$\begin{aligned} m(c, a + b) &= m(c, wx + wy) = [m(c, w)](x + y) = m(c, w)x + m(c, w)y \\ &= m(c, wx) + m(c, wy) = m(c, a) + m(c, b). \end{aligned}$$

In an analogous way, the equations

$$m(a, \alpha b) = \alpha m(a, b) \quad \text{and} \quad m(a + b, c) = m(a, c) + m(b, c)$$

are validated. Hence, m is bilinear.

Let a be in A and $\{x_n\}$ a sequence in A with limit x. Then $\{x_n - x\}$ converges to 0 and the Hewitt-Cohen factorization theorem yields a sequence $\{z_n\}$ and an element z of A such that $\lim_n \|z_n\| = 0$ and $x_n - x = zz_n$ for all $n \in N$. Then

$$\begin{aligned} \overline{\lim_n} \|m(a, x) - m(a, x_n)\| &= \overline{\lim_n} \|m(a, x - x_n)\| = \overline{\lim_n} \|m(a, zz_n)\| \\ &= \overline{\lim_n} \|m(a, z)z_n\| = 0. \end{aligned}$$

It is shown analogously that $\lim_n m(x_n, a) = m(x, a)$. Hence m is separately continuous.

Now let $\{x_n\}$ and $\{y_n\}$ be sequences in A with limits x and y respectively. Invoking the factorization theorem one more time, choose sequences $\{z_n\}$ and $\{w_n\}$ and elements z and w in A such that $\lim \|z_n\| = \lim \|w_n\| = 0$ and, for each $n \in N$, $z_n z = x_n - x$ and $ww_n = y_n - y$. Then

$$\overline{\lim_n}\, \|m(x_n, y_n) - m(x, y)\|$$
$$= \overline{\lim_n}\, \|m(x_n - x, y_n - y) + m(x, y_n) + m(x_n, y) - 2m(x, y)\|$$
$$\leqslant \overline{\lim_n}\, \|m(x_n - x, y_n - y)\| + \overline{\lim_n}\, \|m(x, y_n) - m(x, y)\|$$
$$+ \overline{\lim_n}\, \|m(x_n, y) - m(x, y)\|$$
$$= \overline{\lim_n}\, \|z_n m(z, w) w_n\| + 0 + 0 = 0.$$

Hence, m is jointly continuous. Q.E.D.

The set of all quasi-multipliers on A will be denoted $QM(A)$. Under the usual pointwise operations, $QM(A)$ is a linear space. For each $m \in QM(A)$, let $|||m|||$ be the number $\sup\{\|m(a, b)\| : a, b \in A, \|a\| = \|b\| = 1\}$.

THEOREM 2. *The pair* $(QM(A), |||\ |||)$ *constitutes a Banach space.*

PROOF. For m, $q \in QM(A)$ and $\varepsilon > 0$, choose $a, b \in A$ such that $\|a\| = \|b\| = 1$ and $\|(m + q)(a, b)\| \geqslant |||m + q||| - \varepsilon$. Then

$$|||m||| + |||q||| \geqslant \|m(a, b)\| + \|q(a, b)\| \geqslant \|(m + q)(a, b)\|$$
$$\geqslant |||m + q||| - \varepsilon.$$

This proves the triangle inequality, the only nontrivial element of the proof that $|||\ |||$ is a norm on $QM(A)$.

Now let $\{m_n\}$ be a Cauchy sequence in the normed space $QM(A)$. For $a, b \in A$, the fact

$$\overline{\lim_{n,t}}\, \|m_n(a, b) - m_t(a, b)\| \leqslant \overline{\lim_{n,t}}\, |||m_n - m_t|||\, \|a\|\, \|b\| = 0,$$

with the fact that A is complete, justifies the definition of $m|A \times A \to A$:

$$m(a, b) \equiv \lim_n m_n(a, b).$$

A simple limit argument shows that m satisfies (1). Thus, m is in $QM(A)$. Let ε be a positive number and choose $n \in N$ such that $|||m_s - m_t||| < \varepsilon/2$ for all $s, t \in N$ for which $s, t > n$. Let a and b be elements of A such that $\|a\| = \|b\| = 1$ and choose $t \in N$ such that $t > n$ and $\|m_t(a, b) - m(a, b)\| < \varepsilon/2$. Then, for all $s > n$,

$$\|m_s(a, b) - m(a, b)\| \leqslant \|m_s(a, b) - m_t(a, b)\| + \|m_t(a, b) - m(a, b)\|$$
$$< |||m_s - m_t||| + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since n was chosen independently of a and b, this implies that $|||m_s - m||| < \varepsilon$ for all $s > n$ in N. Hence, $QM(A)$ is complete. Q.E.D.

Define $\phi|A \to QM(A)$ by, for each $a \in A$, letting $[\phi(a)](x, y) \equiv xay$ for all $x, y \in A$. That $\|a\| \geqslant |||\phi(a)|||$ for all $a \in A$ is trivial. To show the opposite inequality, we shall need the following.

LEMMA 1. *If $\{x_\alpha\}$ is an approximate identity for A, then* $\lim_\alpha \|x_\alpha a x_\alpha - a\| = 0$ *for all* $a \in A$.

PROOF. Choose $x, y \in A$ such that $a = xy$. Then

$$\overline{\lim_\alpha} \|x_\alpha a x_\alpha - a\| = \overline{\lim_\alpha} \|x_\alpha xyx_\alpha - x_\alpha xy + x_\alpha xy - xy\|$$
$$\leqslant \overline{\lim_\alpha} \|x_\alpha x\| \, \|yx_\alpha - y\| + \overline{\lim_\alpha} \|x_\alpha xy - xy\| = \|x\| \cdot 0 + 0. \quad \text{Q.E.D.}$$

THEOREM 3. *The map ϕ is a linear isometry of A into $QM(A)$.*

PROOF. That part of the assertion which is neither trivial nor heretofore proved is that $|||\phi_a||| \geqslant \|a\|$ for all $a \in A$. For $a \in A$ and any minimal approximate identity $\{x_\alpha\}$ in A, Lemma 1 yields

$$|||\phi_a||| \geqslant \overline{\lim_\alpha} \|[\phi_a](x_\alpha, x_\alpha)\| = \overline{\lim_\alpha} \|x_\alpha a x_\alpha\| = \|a\|. \quad \text{Q.E.D.}$$

In the sequel, A will be identified with the subspace $\phi(A)$ of $QM(A)$. There are several intermediate subspaces of note. Write $LM(A)$ and $RM(A)$ for the Banach algebras of left and right multipliers, respectively, on A. Define $\lambda | LM(A) \to QM(A)$ by, for each $T \in LM(A)$, letting $[\lambda(T)](x, y) \equiv xT(y)$ for all $x, y \in A$.

THEOREM 4. *The map λ is a linear isometry of $LM(A)$ into $QM(A)$.*

PROOF. That λ is linear is evident. Let T be in $LM(A)$. On the one hand,

$$|||\lambda(T)||| = \sup\{\|aT(b)\| : \|a\| = \|b\| = 1\}$$
$$\leqslant \sup\{\|T(b)\| : \|b\| = 1\} = \|T\|.$$

On the other, if $\varepsilon > 0$, $\{x_\alpha\}$ is a minimal approximate identity in A, and $a \in A$ is a unit vector such that $\|T\| - \varepsilon < \|T(a)\|$, then

$$|||\lambda(T)||| \geqslant \overline{\lim_\alpha} \|[\lambda(T)](x_\alpha, a)\|$$
$$= \overline{\lim_\alpha} \|x_\alpha T(a)\| = \|T(a)\| > \|T\| - \varepsilon.$$

Hence, $|||\lambda(T)||| = \|T\|$. Q.E.D.

The function $\rho | RM(A) \to QM(A)$, defined by $[\rho(T)](x, y) \equiv T(x)y$ for all $x, y \in A$, is also a linear isometry. In the sequel, $LM(Q)$ and $RM(Q)$ will be identified with the subspaces $\lambda(LM(Q))$ and $\rho(RM(Q))$, respectively.

For each pair $(a, b) \in A \times A$, let ${}_a\| \ \|_b$ be the seminorm on $QM(A)$ defined by

$${}_a\|m\|_b \equiv \|m(a, b)\| \quad \text{for all } m \in QM(A).$$

Such a seminorm will be called a *quasi-norm*. The topology $q(A)$ on

$QM(A)$, which is the coarsest for which each quasi-norm is continuous, will be termed the *quasi-norm topology*.

A subring S of a ring R such that $SRS \subset S$ will be called a *quasi-ideal*. Both left and right ideals are quasi-ideals. If A is a quasi-ideal in some superalgebra B, then there is a linear mapping $\phi_B|B \to QM(A)$ defined by, for each $b \in B$,

$$[\phi_B(b)](x, y) = xby \quad \text{for all } x, y \in A.$$

If B is also a normed algebra of which the norm, restricted to A, is just $\| \ \|$, then evidently ϕ_B is norm nonincreasing. The topology $\{\phi_B^{-1}(\mathcal{O}): \mathcal{O} \in q(A)\}$ will be written $q(B, A)$. A net $\{b_\beta\}$ in B $q(B, A)$-converges to some $b \in B$ precisely when $\lim_\beta \|xb_\beta y - xby\| = 0$ for all $x, y \in A$. Thus, by definition, ϕ_B is $q(B, A) - q(A)$ continuous as well as norm continuous. Note that ϕ_A is a $q(A, A) - q(A)$ homeomorphism onto its image.

THEOREM 5. *For each $m \in QM(A)$ and minimal approximate identity $\{x_\alpha\}$ in A,*

$$\lim_\alpha m(x_\alpha, x_\alpha) = m \quad \textit{in the topology } q(A). \tag{3}$$

In particular, the unit ball of A is $q(A)$-dense in the unit ball of $QM(A)$.

PROOF. Let a and b be elements of A. Then

$$\lim_\alpha \|ax_\alpha - a\| = \lim_\alpha \|x_\alpha b - b\| = 0$$

so that, since m is jointly continuous,

$$\begin{aligned} 0 &= \lim_\alpha \|m(ax_\alpha, x_\alpha b) - m(a, b)\| \\ &= \lim_\alpha \|am(x_\alpha, x_\alpha)b - m(a, b)\| = \lim_\alpha {}_a\|m(x_\alpha, x_\alpha) - m\|_b. \end{aligned}$$

Thus $\{m(x_\alpha, x_\alpha)\}$ $q(A)$-converges to m. If $\|m\| \leq 1$, then each $m(x_\alpha, x_\alpha)$ is in the unit ball of A. Q.E.D.

THEOREM 6. *The unit ball of $QM(A)$, as well as $QM(A)$ itself, are $q(A)$-complete.*

PROOF. Let $\{m_\alpha\}$ be a $q(A)$-Cauchy net in $QM(A)$. For all $a, b \in A$, the net $\{m_\alpha(a, b)\}$ is then $\| \ \|$-Cauchy; write $m(a, b)$ for its limit. The function $m|A \times A \to A$ thus defined is evidently a quasi-multiplier and a $q(A)$-limit of $\{m_\alpha\}$. Hence, $QM(A)$ is $q(A)$-complete. If $\{m_\alpha\}$ is in the unit ball of $QM(A)$ and $a, b \in A$ are unit vectors, then

$$\|m(a, b)\| = {}_a\|m\|_b = \lim_\alpha {}_a\|m_\alpha\|_b \leq 1.$$

Hence, $|||m||| \leq 1$. Q.E.D.

Theorems 5 and 6 show that $QM(A)$ may be regarded as the $q(A, A)$-completion of A. Since $q(A, A)$ does not really depend on $QM(A)$, this means that $QM(A)$ might have been defined topologically rather than algebraically. The author chose the latter approach, mainly, because it extends to arbitrary (nonnormed) rings. The subspaces $LM(A)$ and $RM(A)$ of $QM(A)$ are also completions of A, but of slightly (sometimes) finer topologies (see [10, 1.12], for instance). The following theorem is useful for applications.

THEOREM 7. *Let B be a Banach algebra in which A is a quasi-ideal (and sub-Banach algebra). Then ϕ_B is a linear isometry of B onto $QM(A)$ if and only if the following conditions hold*:

(i) *the unit ball B_1 of B is $q(B, A)$-complete*;

(ii) $\|b\| = \sup\{\|abc\|: a, c \in A_1 = \text{unit ball of } A\}$ *for each $b \in B$.*

PROOF. That the conditions are necessary follow from Theorem 6 and the definition of $||| \ |||$.

Suppose that B satisfies (i) and (ii). Condition (ii) implies that ϕ_B is an isometry. We need only show that ϕ_B is surjective; in fact, it will suffice to show that $\phi_B(B)$ contains the boundary of the unit ball of $QM(A)$. Let then $m \in QM(A)$ be a unit vector. Theorem 5 yields a net $\{y_\beta\}$ in A_1 with $q(A)$-limit m. Then $\{y_\beta\}$ is $q(A, A)$-Cauchy, hence $q(B, A)$-Cauchy, and, by (i), has a $q(B, A)$-limit b in B_1. For each $w, z \in A$,

$$\begin{aligned} &\|m(w, z) - [\phi_B(b)](w, z)\| \\ &\qquad \leq \overline{\lim_\beta} \|m(w, z) - wy_\beta z\| + \overline{\lim_\beta} \|wy_\beta z - [\phi_B(b)](w, z)\| \\ &\qquad = \overline{\lim_\beta} {}_w\|m - y_\beta\|_z + \overline{\lim_\beta} \|wy_\beta z - wbz\| = 0. \end{aligned}$$

That is, $m = \phi_B(b)$. Hence, ϕ_B is surjective. Q.E.D.

If A has an identity u and m is a quasi-multiplier, then, for all $x, y \in A$,

$$[\phi(m(u, u))](x, y) = xm(u, u)y = m(x, y).$$

Thus, in this case, A may be identified with $Q(A)$.

An approximate identity $\{x_\alpha\}$ in A will be called an *ultra-approximate identity* for A if, for all $m \in QM(A)$ and $a \in A$, the nets $\{m(a, x_\alpha)\}$ and $\{m(x_\alpha, a)\}$ are $\| \ \|$-Cauchy.

THEOREM 8. *An approximate identity $\{x_\alpha\}$ in A is an ultra-approximate identity if and only if, for all $a \in A$, $T \in LM(A)$, and $S \in RM(A)$, $\{aT(x_\alpha)\}$ and $\{S(x_\alpha)a\}$ are $\| \ \|$-Cauchy.*

PROOF. First assume that $\{x_\alpha\}$ is an ultra-approximate identity and consider arbitrary $a \in A$, $T \in LM(A)$, and $S \in RM(A)$. Then $\{aT(x_\alpha)\} = \{[\lambda(T)](a, x_\alpha)\}$ and $\{S(x_\alpha)a\} = \{[\rho(S)](x_\alpha, a)\}$ so that all are Cauchy.

Now assume that $\{aT(x_\alpha)\}$ and $\{S(x_\alpha)a\}$ are $\| \ \|$-Cauchy for all $a \in A$,

$T \in LM(A)$, and $S \in RM(A)$. Let m be a quasi-multiplier and a an element of A. Choose $b, c \in A$ such that $a = bc$. Define $T|A \to A$ and $S|A \to A$ by letting

$$T(x) \equiv m(c, x) \text{ and } S(x) \equiv m(x, b)$$

for all $x \in A$. Then T is in $LM(A)$ and S in $RM(A)$. But, for each index α,

$$m(a, x_\alpha) = bm(c, x_\alpha) = bT(x_\alpha)$$

and

$$m(x_\alpha, a) = m(x_\alpha, b)c = S(x_\alpha)c.$$

Hence, $\{m(x_\alpha, a)\}$ and $\{m(a, x_\alpha)\}$ are $\| \, \|$-Cauchy. Q.E.D.

Lemma 2. *Let* $\{x_\alpha\}_{\alpha \in \Gamma}$ *be an ultra-approximate identity for* A. *Let* $m, n \in RM(A)$ *and* $a \in A$. *Then*

$$\lim_{\beta \in \Gamma} m(x_\beta) \lim_{\alpha \in \Gamma} n(x_\alpha)a = \lim_{\delta \in \Gamma} nm(x_\delta)a.$$

Proof. Let $\varepsilon > 0$. Write d for $\lim_\delta nm(x_\delta)a$, e for $\lim_\alpha n(x_\alpha)a$, and b for $\lim_\beta m(x_\beta)e$. Choose $\sigma \in \Gamma$ such that

$$\|m(x_\sigma)e - b\| < \varepsilon/4 \text{ and } \|nm(x_\sigma)a - d\| < \varepsilon/4.$$

Choose $\gamma \in \Gamma$ such that $\|n(x_\gamma)a - e\| \leqslant \varepsilon/4\|m\|$, and

$$\|m(x_\sigma)x_\gamma - m(x_\sigma)\| < \varepsilon/(4\|n\| \, \|a\|).$$

Then

$$\begin{aligned} \|b - d\| &\leqslant \|m(x_\sigma)e - b\| + \|m(x_\sigma)n(x_\gamma)a - m(x_\sigma)e\| \\ &\quad + \|n(m(x_\sigma)x_\gamma)a - n(m(x_\sigma))a\| + \|nm(x_\sigma)a - d\| \\ &\leqslant \varepsilon/4 + \|m(x_\sigma)\| \, \|n(x_\gamma)a - e\| + \|n\| \, \|m(x_\sigma)x_\gamma - m(x_\sigma)\| \, \|a\| + \varepsilon/4 \\ &\leqslant \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon. \end{aligned}$$

Q.E.D.

Theorem 9. *Let* A *possess an ultra-approximate identity. Then the maps* $\lambda|LM(A) \to QM(A)$ *and* $\rho|RM(A) \to QM(A)$ *are surjective. Further, the maps* $\lambda^{-1} \circ \rho$ *and* $\rho^{-1} \circ \lambda$ *are isometric algebra anti-isomorphisms.*

Proof. Let $\{x_\alpha\}$ be an ultra-approximate identity. Let q be in $QM(A)$. Define $m|A \to A$ by letting, for each $a \in A$,

$$m(a) \equiv \lim_\alpha q(x_\alpha, a).$$

If $a, b \in A$, then

$$m(ab) = \lim_\alpha q(x_\alpha, ab) = \lim_\alpha q(x_\alpha, a)b = m(a)b.$$

It follows that $m \in LM(A)$. For all $a, b \in A$,

$$\lambda_m(a, b) = am(b) = \lim_\alpha aq(x_\alpha, b) = \lim_\alpha q(ax_\alpha, b) = q(a, b).$$

Thus $\lambda_m = q$ and λ is surjective. That ρ is surjective is proved analogously.

Now consider $m, n \in RM(A)$ and $a \in A$. We have, by Lemma 2,

$$\begin{aligned}\lambda^{-1}(\rho_m)\lambda^{-1}(\rho_n)(a) &= \lambda^{-1}(\rho_m)\lim_\alpha \rho_n(x_\alpha, a)\\ &= \lambda^{-1}(\rho_m)\lim_\alpha n(x_\alpha)a = \lim_\beta m(x_\beta)\lim_\alpha n(x_\alpha)a\\ &= \lim_\zeta n \circ m(x_\zeta)a = \lambda^{-1}(\rho_{nm})(a).\end{aligned}$$

It is now evident that $\lambda^{-1} \circ \rho$ is an algebra anti-isomorphism. That $\rho^{-1} \circ \lambda$ is as well is proved analogously.

Since ρ and λ are isometries, so are $\lambda^{-1} \circ \rho$ and $\rho^{-1} \circ \lambda$. Q.E.D.

Thus, when A possesses an ultra-approximate identity $\{x_\alpha\}$, $QM(A)$ may be made into an algebra in two different ways, either making λ or ρ an isomorphism. We record for reference that, for $q \in QM(A)$ and $a \in A$,

(4) $$\lim_\alpha q(x_\alpha, a) = [\lambda^{-1}(q)](a);$$

(5) $$\lim_\alpha q(a, x_\alpha) = [\rho^{-1}(q)](a).$$

The converse to Theorem 9 is also true.

THEOREM 10. *If λ and ρ are surjective and $\{x_\alpha\}$ is an approximate identity, then it is ultra-approximate.*

PROOF. If $q \in QM(A)$ and $a \in A$, then

$$[\lambda^{-1}(q)](a) = \lim_\alpha x_\alpha[\lambda^{-1}(q)](a) = \lim_\alpha q(x_\alpha, a).$$

Similarly, $[\rho^{-1}(q)](a) = \lim_\alpha q(a, x_\alpha)$. Q.E.D.

Let now B be a Banach algebra in which A is a quasi-ideal and such that ϕ_B is an isometry; such a B will be called an *intermediate algebra for A*. It follows from (4) and (5) that, if $b \in B$, $a \in A$, and $\{x_\alpha\}$ is ultra-approximate, then

(6) $$\lim_\alpha \|x_\alpha ba - ba\| = 0;$$

(7) $$\lim_\alpha \|abx_\alpha - ab\| = 0.$$

THEOREM 11. *Let B be an intermediate algebra for A and suppose that A possesses a bounded ultra-approximate identity. Then ϕ_B is an algebra homomorphism.*

PROOF. Let $\{x_\gamma\}$ be an ultra-approximate identity for A. Consider $a, b \in B$, $x, y \in A$, and $\varepsilon > 0$. Choose an index γ_0 such that, for all $\gamma > \gamma_0$,

$$\|[\phi_B(a)\cdot\phi_B(b)](x, y) - [\phi_B(a)](x, x_\gamma)[\phi_B(b)](x_\gamma, y)\| < \varepsilon.$$

Invoke (6) to obtain an index $\gamma > \gamma_0$ such that

$$\|x_\gamma by - by\| < \varepsilon.$$

Thus, for this γ,

$$\begin{aligned}&\|[\phi_B(ab)](x,y)-[\phi_B(a)\cdot\phi_B(b)](x,y)\|\\&\quad\leqslant\|[\phi_B(ab)](x,y)-xax_\gamma by\|+\|xax_\gamma by-xax_\gamma x_\gamma by\|\\&\qquad+\|[\phi_B(a)](x,x_\gamma)\cdot[\phi_B(b)](x_\gamma,y)-[\phi_B(a)\cdot\phi_B(b)](x,y)\|\\&\quad\leqslant\|xa\|\,\|by-x_\gamma by\|+\|xax_\gamma\|\,\|by-x_\gamma by\|+\varepsilon\\&\quad\leqslant\varepsilon\|x\|\,\|a\|+\|x\|\,\|a\|\,\|x_\gamma\|\varepsilon+\varepsilon.\end{aligned}$$

Since ε, x, and y were chosen arbitrarily, $\phi_B(ab)=\phi_B(a)\cdot\phi_B(b)$. Q.E.D.

We will henceforth regard intermediate algebras of A as subalgebras of $QM(A)$, whenever A possesses a bounded ultra-approximate identity.

THEOREM 12. *Let B and A be as in Theorem 11. Then, for each $m\in QM(B)$, $m(A\times A)\subset A$. Further, if $m\in QM(B)$ vanishes on $A\times A$, then $m=0$. Thus, $QM(B)$ may be identified with a linear subspace of $QM(A)$.*

PROOF. Let $\{x_\alpha\}$ be a bounded, ultra-approximate identity for A. Let m be in $QM(B)$ and $x,y\in A$. Choose $a,b,c,d\in A$ such that $x=ab$ and $y=cd$. Lemma 1 implies

$$0=\lim_\alpha\|x_\alpha am(b,c)dx_\alpha-am(b,c)d\|=\lim_\alpha\|x_\alpha am(b,c)dx_\alpha-m(x,y)\|.$$

Since A is a quasi-ideal of B and A is complete, it follows that $m(x,y)\in A$. This means that $m(A\times A)\subset A$.

Now suppose that $m\in QM(B)$ vanishes on $A\times A$. Assume that $m\neq 0$ and choose $a,b\in B$ such that $m(a,b)\neq 0$. Then there exists $x,y\in A$ such that $xm(a,b)y\neq 0$. By (6) and (7)

$$\lim_\alpha\|xax_\alpha-xa\|=0=\lim_\alpha\|x_\alpha by-by\|.$$

Let ε be any positive number. Since m is separately continuous, there exists an index α_0 such that

$$\|m(xa,by)-m(xax_{\alpha_0},by)\|<\varepsilon.$$

Now choose an index α such that

$$\|m(xax_{\alpha_0},by)-m(xax_{\alpha_0},x_\alpha by)\|<\varepsilon.$$

It follows that

$$0\neq\|xm(a,b)y\|=\|m(xa,by)\|\leqslant\|m(xax_{\alpha_0},x_\alpha by)\|+2\varepsilon.$$

Since ε was arbitrarily chosen and both xax_{α_0} and $x_\alpha by$ are in A, it follows that m is nonzero on $A\times A$. Q.E.D.

Thus, if A has a bounded ultra-approximate identity and B is an intermediate algebra of A, then $QM(B)$ may be regarded as a subset of $QM(A)$.

For the remainder of this section, our attention will primarily be fixed not upon A itself, but on a subalgebra $\underline{A}$ of A which is realized as a quasi-ideal of a rather concrete Banach algebra $\underline{B}$.

Let $\underline{B}$ be an algebra which is also a dense subspace of a Banach space $(E, \| \|_E)$ such that, for all $b \in \underline{B}$,

$$(8) \qquad \|b\|_B \equiv \sup\{\max\{\|bx\|_E, \|xb\|_E\}: x \in E_1 \cap \underline{B}\} < \infty$$

(where E_1 is the unit ball of E). It is evident that $(\underline{B}, \| \|_B)$ is a normed algebra and that the multiplicative operation extends in exactly one way to a continuous linear map from $(B \times E) \cup (E \times B)$ to E, where B denotes the Banach algebra completion of E. We shall say that $\underline{B}$ is an *algebra compatible with E.* If

$$(9) \qquad \sup\{\max\{\|ax\|_E, \|xa\|_E\}: x \in E_1 \cap \underline{D}\} < \infty \quad \text{for some } \| \|_E\text{-dense subset } \underline{D} \text{ of } \underline{B},$$

occurs only for those $a \in E$ which lie in B, we shall say that B is $\underline{D}$-maximal

THEOREM 13. *Let $\underline{B}$ be a $\underline{B}$-maximal algebra compatible with a Banach space E. Then $\underline{B}$ is a Banach algebra under the norm $\| \|_E + \| \|_B$.*

PROOF. For all $a, b \in \underline{B}$, we have

$$\|ab\|_E + \|ab\|_B \leq \|a\|_B \|b\|_E + \|a\|_B \|b\|_B \leq (\|a\|_B + \|a\|_E)(\|b\|_B + \|b\|_E)$$

so that $\underline{B}$ is a normed algebra under $\| \|_E + \| \|_B$.

Let $\{b_n\}$ be a $\| \|_E + \| \|_B$-Cauchy sequence in $\underline{B}$. Then $\{b_n\}$ has a $\| \|_E$-limit b in E. For each $x \in E_1 \cap \underline{B}$,

$$\|bx\|_E = \lim_n \|b_n x\|_E \leq \sup_n \|b_n\|_B \|x\|_E \leq \sup_n \|b_n\|_B$$

and, similarly,

$$\|xb\|_E \leq \sup \|b_n\|_B.$$

Thus b satisfies (9) and so is in $\underline{B}$. Hence $\underline{B}$ is $\| \|_E + \| \|_B$-complete. Q.E.D.

Throughout the remainder of §2, $\underline{B}$ will be an $\underline{A}$-maximal algebra compatible with a Banach space $(E, \| \|_E)$, the norm $\| \|_B$ will be as in (8), the $\| \|_B$-completion of $\underline{B}$ will be denoted by B, and we shall write $\| \|_{\underline{B}}$ for the norm $\frac{1}{2}(\| \|_E + \| \|_B)$. Furthermore, $\underline{A}$ will be an algebra such that

$$(10) \qquad \underline{A} \text{ is a } \| \|_{\underline{B}}\text{-closed quasi-ideal in } \underline{B};$$

$$(11) \qquad \underline{A}\underline{B}\underline{A} \text{ is } \| \|_E\text{-dense in } E.$$

Finally, $(\underline{A}, \| \|_B)$ will be assumed to possess a minimal approximate identity $\{x_\alpha\}$ such that

$$(12) \quad \lim_\alpha \sup\{\max\{\|x_\alpha ba - ba\|_{\underline{B}}, \|abx_\alpha - ab\|_{\underline{B}}\}: b \in \underline{B}, \|b\|_B < 1\} = 0$$

for all $a \in \underline{A}$. The closure of $\underline{A}$ in B will be denoted A.

From (12), it follows that

(13) $\lim_\alpha \sup\{\max\{\|x_\alpha ba - ba\|_B, \|abx_\alpha - ab\|_B\}: b \in B, \|b\|_B \leqslant 1\} = 0$

for all $a \in A$. We also have

(14) $$\lim_\alpha \|x_\alpha a - a\|_B = 0 = \lim_\alpha \|ax_\alpha - a\|_B \quad \text{for all } a \in A.$$

From (11) follows that AE and EA are $\| \;\|_E$-dense in E; consequently, Hewitt's Factorization Theorem for Banach modules yields

(15) $$E = AE = EA = AEA.$$

This, with (14), yields

(16) $$\lim_\alpha \|x_\alpha y - y\|_E = 0 = \lim_\alpha \|yx_\alpha - y\|_E \quad \text{for all } y \in E.$$

Together, (14) and (16) yield

(17) $$\lim_\alpha \|x_\alpha a - a\|_{\underline{B}} = 0 = \lim_\alpha \|ax_\alpha - a\|_{\underline{B}} \quad \text{for all } a \in \underline{A}.$$

THEOREM 14. *The equalities $\underline{A} = A\underline{A} = \underline{A}A = A\underline{A}A$ obtain. In particular, $\underline{A}$ is an ideal of A.*

PROOF. Let $x \in \underline{A}$ and $a \in A$. Then $ax \in E$ and, since (9) holds for ax, $ax \in \underline{B}$ as well. But (14) and (16) imply that

$$\lim_\alpha \|x_\alpha axx_\alpha - ax\|_{\underline{B}} \leqslant \lim_\alpha \|x_\alpha ax - ax\|_{\underline{B}} \|x_\alpha\|_B + \lim_\alpha \|a\|_B \|xx_\alpha - x\|_{\underline{B}} = 0.$$

By (10), this yields $ax \in \underline{A}$. We have shown that $A\underline{A} \subset \underline{A}$. That $A\underline{A}$ is $\| \;\|_{\underline{B}}$-dense in $\underline{A}$ follows from (17); hence, Hewitt's Factorization Theorem implies $A\underline{A} = \underline{A}$. That $\underline{A}A = \underline{A}$ is proved analogously. Q.E.D.

From (10) and (13), it follows that

(18) $$\underline{A} \text{ is an ideal in } \underline{B}.$$

This yields

(19) $$A \text{ is an ideal in } B.$$

For all $a \in A \cap \underline{B}$, from (14) and (16) follow $\lim_\alpha \|x_\alpha a - a\|_{\underline{B}} = 0$; thus, (10) and (18) yield $a \in \underline{A}$:

(20) $$A \cap \underline{B} = \underline{A}.$$

For a subset $\underline{D}$ of B, we shall write $QM(E; \underline{D})$ for the set of all bilinear, separately continuous functions $m | E \times E \to E$ such that

$$m(ax, yb) = am(x, y)b \quad \text{for all } x, y \in E \text{ and } a, b \in \underline{D}.$$

A simple limit argument shows that

(21) $$QM(E; \underline{D}) = QM(E; D)$$

where D is the $\| \;\|_B$-closed linear span of $\underline{D}$. The arguments of Theorem 1 may be employed to show that

(22) $$m(x, y)a = m(x, ya) \quad \text{and} \quad am(x, y) = m(ax, y)$$

for all $x, y \in E$, $a \in A$, and $m \in QM(E; A)$, and that m is jointly continuous; consequently

(23) $$|||m|||_E = \sup\{\|m(x,y)\|_E : x, y \in E_1\} < \infty$$

(where E_1 is the $\|\ \|_E$-unit ball of E). It is easy to see that $QM(E; A)$ is a Banach space under $|||\ |||_E$.

If $\underline{D}$ is a subalgebra of B, we shall write simply $QM(\underline{D})$ for $QM((\underline{D}, \|\ \|_B))$. It is evident that if D is the $\|\ \|_B$-closure of $\underline{D}$, then each $m \in QM(\underline{D})$ can be extended in precisely one way to an element of $QM(D)$; thus, we may regard $QM(\underline{D})$ as a subspace of $QM(D)$. For $m \in QM(\underline{D})$, we write

$$|||m|||_D = \sup\{\|m(a,b)\|_B : a, b \in \underline{D};\ \|a\|_B, \|b\|_B\} \leqslant 1$$

(which may possibly be infinity).

THEOREM 15. *For all* $m \in QM(E; A)$, *we have*

$$m|_{\underline{B}\times\underline{B}} \in QM((\underline{B}, \|\ \|_{\underline{B}})) \quad \textit{and} \quad m|_{\underline{A}\times\underline{A}} \in Q(\underline{A}).$$

PROOF. For all $x \in \underline{A}$ and $a, b \in \underline{B}$, we have

$$\|m(a,b)x\|_E = \|m(a,bx)\|_E \leqslant |||m|||_E\, \|a\|_E\, \|bx\|_E \leqslant |||m|||_E\, \|a\|_E\, \|b\|_B\, \|x\|_E$$

and, similarly,

$$\|xm(a,b)\|_E \leqslant |||m|||_E\, \|a\|_B\, \|b\|_E\, \|x\|_E.$$

This means that (9) is satisfied for $m(a, b)$; hence, $m(a, b) \in \underline{B}$. That $m|_{B\times B}$ is separately continuous is a simple consequence of the Closed Graph Theorem. Thus, $m|_{B\times B} \in QM((\underline{B}, \|\ \|_B))$.

Let $x, y \in \underline{A}$. Theorem 15 permits choice of $c \in A$ and $d \in \underline{A}$ such that $x = cd$. It follows from the above that $m(x, y), m(d, y) \in \underline{B}$ and so, from (19), that $m(x, y) = cm(d, y) \in A$. But now (20) implies $m(x, y) \in \underline{A}$.

Now suppose that $x_0, y \in \underline{A}$ and that $\{x_n\}$ is a sequence in $\underline{A}$ with $\|\ \|_B$-limit x_0. Let $\varepsilon > 0$. By Theorem 15, there exist $a \in \underline{A}$ and $b \in A$ such that $ab = y$. From (13) we have

$$\lim_\alpha \sup_{n=0}^{\infty} \|x_\alpha m(x_n, y) - m(x_n, y)\|_B$$
$$= \lim_\alpha \sup_{n=0}^{\infty} \|x_\alpha m(x_n, a)b - m(x_n, a)b\|_B = 0.$$

Choose an index α such that

$$\|x_\alpha m(x_n, y) - m(x_n, y)\|_B < \varepsilon/2$$

for all $n = 0, 1, \ldots$. We have

$$\overline{\lim}\, \|m(x_n, y) - m(x_0, y)\|_B$$
$$\leqslant \overline{\lim}\Big(\|m(x_n, y) - x_\alpha m(x_n, y)\|_B$$
$$+ \|m(x_\alpha[x_n - x_0], y)\|_B + \|x_\alpha m(x_0, y) - m(x_0, y)\|_B\Big)$$
$$\leqslant \varepsilon/2 + \overline{\lim_n}\ |||m|||_E \|x_\alpha\|_E\, \|x_n - x_0\|_B\, \|y\|_E + \varepsilon/2 = \varepsilon.$$

Thus, $\lim_n \|m(x_n, y) - m(x_0, y)\|_B = 0$. That $\lim_n \|m(y, x_n) - m(y, x_0)\|_B = 0$ is proved analogously. Hence, $m \in QM(A)$. Q.E.D.

THEOREM 16. *The natural restriction mapping on $QM(A)$ is a linear isometry onto $QM(\underline{A})$.*

PROOF. Let $m \in QM(A)$ and $a, b \in \underline{A}$. By Theorem 14, there exist $c \in A$ and $d \in \underline{A}$ such that $a = dc$. Thus,

$$m(a, b) = m(dc, b) = dm(c, b)$$

which, by Theorem 14, is in $\underline{A}$. Thus, $m|_{\underline{A}\times\underline{A}} \in QM(\underline{A})$. That this restriction mapping is an isometry is evident. Q.E.D.

THEOREM 17. *The natural restriction mapping (to $\underline{A} \times \underline{A}$) on $QM(E; \underline{A})$ is a topological isomorphism onto $QM(\underline{A})$.*

PROOF. That the mapping in question is into $QM(\underline{A})$ follows from (21) and Theorem 15. Let $m \in QM(\underline{A})$ and consider $a, b \in \underline{A}$. From Theorem 14, there exist $c \in A$ and $d \in \underline{A}$ such that $b = dc$. We have, by (14) and (8),

$$\|m(a, b)\|_E = \|m(a, d)c\|_E = \lim_\alpha \|m(ax_\alpha, d)c\|_E$$
$$= \lim_\alpha \|am(x_\alpha, b)\|_E \leq \|a\|_E \, |||m|||_A \, \|b\|_B$$

since $\|x_\alpha\|_B \leq 1$ for each α. Thus m is right separately $\| \; \|_E$-continuous; that it is left separately continuous is proved analogously. Thus there exists precisely one separately continuous extension $\bar{m}|E \times E \to E$. That $\bar{m} \in QM(E; \underline{A})$ is evident. Thus the mapping ϕ restricting elements of $QM(E; \underline{A})$ to $\underline{A} \times \underline{A}$ is surjective.

It is a consequence of Theorem 16 that $QM(\underline{A})$ is complete. Since $QM(E; A)$ is also complete, a simple application of the Closed Graph Theorem shows that ϕ is continuous. That ϕ is a homeomorphism now follows from the Open Mapping Theorem. Q.E.D.

THEOREM 18. *Let $\phi|QM(E; \underline{A}) \to E^{\underline{B}\times\underline{B}}$ and $\psi|QM(\underline{B}) \to \underline{B}^{\underline{A}\times\underline{A}}$ be the natural restriction mappings. Then ϕ is a topological isomorphism onto $QM(\underline{B})$ and ψ a linear isometry onto $QM(\underline{A})$.*

PROOF. That ϕ is into $QM(\underline{B})$ follows from Theorem 15. Let $m \in QM(\underline{B})$ and $a, b \in \underline{A}$. By (14), we have

$$\lim_\alpha \|m(a, b) - x_\alpha m(a, b)\|_B = \lim_\alpha \|m(a - x_\alpha a, b)\|_B = 0$$

so that, by (19), $m(a, b) \in A$. But (20) then implies that $m(a, b) \in \underline{A}$. This proves that ψ is into $QM(\underline{A})$.

By Theorem 17, there exists precisely one element $\bar{m}$ of $Q(E; \underline{A})$ such that $\bar{m}|_{\underline{A}\times\underline{A}} = \psi(m)$. Let $\varepsilon > 0$ and $a, b \in \underline{B}$. By (8), there exists $x, y \in E_1$ such that either $\|m(a, b)x\|_E \geq \|m(a, b)\|_B - \varepsilon$ or $\|xm(a, b)\|_E \geq \|m(a, b)\|_B -$

ε. For definiteness, let us assume the former. By (15), there exist $y \in E$ and $e \in A$ such that $ey = x$; in fact, Hewitt's Factorization Theorem permits us to assume as well that $\|e\|_B \leqslant 1$ and $\|y\|_E < \|x\|_E + \varepsilon$.

We have, by (8),

$$\|\overline{m}(a, b)e\|_B \geqslant \left\|\overline{m}(a, b)e \frac{y}{\|y\|_E}\right\|_E = \|\overline{m}(a, b)x\|_E / \|y\|_E \geqslant \frac{\|\overline{m}(a, b)\|_B - \varepsilon}{1 + \varepsilon}.$$

This, with (13), yields

$$\begin{aligned} \frac{\|\overline{m}(a, b)\|_B - \varepsilon}{1 + \varepsilon} &\leqslant \|\overline{m}(a, b)e\|_B \\ &= \lim_\alpha \|x_\alpha \overline{m}(a, b)e\|_B = \lim_\alpha \|m(x_\alpha a, be)\| \\ &\leqslant \overline{\lim_\alpha} \, |||\psi(m)|||_A \, \|x_\alpha\|_B \, \|a\|_B \, \|b\|_B \, \|e\|_B \leqslant |||\psi(m)|||_A \, \|a\|_B \, \|b\|_B. \end{aligned}$$

It follows that $\phi(\overline{m}) \in QM(\underline{B})$ and $|||\phi(\overline{m})|||_B \leqslant |||\psi(m)|||_A$. That the reverse inequality holds is trivial. Thus

$$(24) \qquad |||\phi(\overline{m})|||_B = |||\psi(m)|||_A \quad \text{for all } m \in QM(A).$$

From Theorem 17, we know that $\psi \circ \phi$ is bijective. It follows that ϕ is injective and ψ is surjective. Thus, if we can show that ψ is injective, we shall know that both ϕ and ψ are bijections. Hence, (24) and Theorem 17 will imply that ψ is an isometry and ϕ a topological isomorphism. Suppose that $q \in QM(\underline{B})$ and assume that $q \neq 0$. Choose $r, t \in \underline{B}$ such that $q(r, t) \neq 0$. Then there exists $x \in E$ such that either $q(r, t)x \neq 0$ or $xq(r, t) \neq 0$; for definiteness, we assume the former. By (11), it follows that, for some $s \in \underline{A}$, $q(r, t)s \neq 0$. Thus, (13) implies

$$0 \neq \|q(r, t)s\|_B = \lim_\alpha \|x_\alpha q(r, t)s\|_B = \lim_\alpha \|q(x_\alpha r, ts)\|_B.$$

So, for some index α, $\|q(x_\alpha r, ts)\|_B \neq 0$. Hence

$$[\psi(q)](x_\alpha r, ts) = q(x_\alpha r, ts) \neq 0.$$

Thus ψ is injective. Q.E.D.

THEOREM 19. *Let $\underline{D}$ be a subalgebra of $\underline{B}$ containing the net $\{x_\alpha\}$ (as in (12)) and such that both $\underline{B}$ is $\underline{D}$-maximal and $D \cap E$ is dense in E. Then $QM(E; \underline{B}) = QM(E; \underline{D})$.*

PROOF. That $QM(E; \underline{B}) \subset QM(E; \underline{D})$ is trivial. Let D be the $\| \, \|_B$-closure of $\underline{D}$ in B and let $m \in QM(E; \underline{D})$. Since D is a Banach algebra with minimal approximate identity and, since (16) implies that DE and ED are dense in E, Hewitt's Factorization Theorem implies

$$E = DE = ED = DED.$$

Further, as in the proof of Theorem 1, it follows that

(25) $m(x,y)d = m(x,yd)$ and $dm(x,y) = m(dx,y)$ for all $x, y \in E$

and that

(26) $$|||m|||_E \equiv \sup\{\|m(a,b)\|_B;\ a, b \in E_1\} < \infty.$$

Let $x \in \underline{D} \cap E_1$ and $a, b \in \underline{B}$. Then

$$\|m(a,b)x\|_E = \|m(a,bx)\|_E \leqslant |||m|||_E \|a\|_E \|b\|_B \|x\|_E \leqslant |||m|||_E \|a\|_E \|b\|_B$$

and, similarly

$$\|xm(a,b)\|_E \leqslant |||m|||_E \|a\|_B \|b\|_E.$$

Hence, (9) is satisfied so that

(27) $$m(a,b) \in \underline{B} \quad \text{for all } a, b \in \underline{B}.$$

Now let $m \in QM(E; D)$, $x, \underline{x} \in E$ and $b, \underline{b} \in \underline{B}$. Let $\varepsilon > 0$ and, in virtue of (11) and (18), choose $a, \underline{a} \in \underline{A}$ such that

$$\|x - a\|_E + \|\underline{x} - \underline{a}\|_E < \frac{\varepsilon}{8|||m|||_E \|b\|_B \|\underline{b}\|_B (\|\underline{x}\|_E + \|x\|_E + \|a\|_E + \|\underline{a}\|_E) + 1}.$$

Choose $d, \underline{d} \in \underline{D}$ such that

$$\|d - b\|_E + \|\underline{d} - \underline{b}\|_E$$
$$< \frac{\varepsilon}{8|||m|||_E \|\underline{a}\|_B (\|b\|_B \|a\|_E + \|\underline{d}\|_E \|a\|_B) + \|m(a, \underline{ad})\|_B + \|bm(a, \underline{a})\|_B + 1}.$$

By (25) and (26), we have

$$\|m(bx, \underline{xb}) - bm(x, \underline{x})\underline{b}\|_E$$
$$\leqslant \|m(bx, \underline{xb}) - m(ba, \underline{xb})\|_E + \|m(ba, \underline{xb}) - m(ba, \underline{ab})\|_E$$
$$+ \|m(ba, \underline{ab}) - m(ba, \underline{ad})\|_E$$
$$+ \|m(ba, \underline{ad}) - m(da, \underline{ad})\|_E + \|dm(a, \underline{ad}) - bm(a, \underline{ad})\|_E$$
$$+ \|bm(a, \underline{a})\underline{d} - bm(a, \underline{a})\underline{b}\|_E$$
$$+ \|bm(a, \underline{a})\underline{b} - bm(x, \underline{a})\underline{b}\|_E + \|bm(x, \underline{a})\underline{b} - bm(x, \underline{x})\underline{b}\|_E$$
$$\leqslant |||m|||_E \|b\|_B \|x - a\|_E \|\underline{x}\|_E \|\underline{b}\|_B + |||m|||_E \|b\|_B \|a\|_E \|\underline{x} - \underline{a}\|_E \|\underline{b}\|_B$$
$$+ |||m|||_E \|b\|_B \|a\|_E \|\underline{a}\|_B \|\underline{b} - \underline{d}\|_E + |||m|||_E \|\underline{a}\|_B \|\underline{d}\|_E \|b - d\|_E \|a\|_B$$
$$+ \|d - b\|_E \|m(a, \underline{ad})\|_B + \|bm(a, \underline{a})\|_B \|\underline{d} - \underline{b}\|_E$$
$$+ \|b\|_B \|\underline{b}\|_B |||m|||_E \|\underline{a}\|_E \|a - x\|_E$$
$$+ \|b\|_B \|\underline{b}\|_B |||m|||_E \|x\|_E \|\underline{a} - \underline{x}\|_E$$
$$< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \varepsilon.$$

This proves that $m \in QM(E; \underline{B})$. Q.E.D.

3. **Quasi-multipliers on group algebras of SIN groups.** Let G be a locally compact, Hausdorff, topological group with equivalent left and right uniform structures (G is a SIN group). Then G is unimodular and so possesses a (two-sided) Haar measure λ. Other unexplained notation will be as in [3] and [4]. Let p be an element of $[1, \infty[$. Consider the following condition for a function $f \in L_p(G)$:

$$(28) \qquad \|f\|_p^{rt} \equiv \sup\{\|g * f\|_p : g \in C_{00}(g); \|g\|_p \leqslant 1\} < \infty.$$

The set of all $f \in L_p(G)$ satisfying (28) will be written $L_p^{rt}(G)$; its elements are called (right-) *p-tempered*. Each (right-) p-tempered function f may be viewed as a $\| \ \|_p$-continuous linear (right multiplication) operator on $C_{00}(G)$. Hence it has a unique $\| \ \|_p$-continuous extension to an operator on $L_p(G)$; its value at a $g \in L_p(G)$ will be written $g * f$. The set $L_p^{rt}(G)$ is closed under this extended operation $*$; under the norm $\| \ \|_p^{rt}$ it is a normed algebra and, under the norm $||| \ |||_p^{rt} \equiv \frac{1}{2}(\| \ \|_p^{rt} + \| \ \|_p)$, it is a Banach algebra. It is also a right Banach $L_1(G)$-module and, by the Hewitt Factorization Theorem, $L_p^{rt}(G) * L_1(G)$ is a $||| \ |||_p^{rt}$-closed subspace of L_p^{rt}; it will be written $L_p^{rwt}(G)$ and its elements will be called *(right-) p-well-tempered.*

The sets $L_p^{lt}(G)$ and $L_p^{lwt}(G)$ of *(left-) p-tempered* and *(left-) p-well-tempered* functions, and the norms $\| \ \|_p^{lt}$ and $||| \ |||_p^{lt}$ are defined analogously.

We shall write $L_p^t(G)$ for $L_p^{lt}(G) \cap L_p^{rt}(G)$ and shall call its elements simply *p-tempered.* Similarly, the elements of $L_p^{wt}(G) = L_p^{lwt}(G) \cap L_p^{rwt}(G)$ are simply *p-well-tempered.* The norm $\| \ \|_p^t \equiv \max\{\| \ \|_p^{rt}, \| \ \|_p^{lt}\}$ renders $L_p^t(G)$ a normed algebra under convolution. The norm $||| \ |||_p^t \equiv \frac{1}{2}(\| \ \|_p^t + \| \ \|_p) = \frac{1}{2}(||| \ |||_p^{rt} + ||| \ |||_p^{lt})$ renders $L_p^t(G)$ into a Banach algebra.

We now set $E \equiv L_p(G)$, $\underline{A} \equiv L_p^{wt}(G)$ and $\underline{B} \equiv L_p^t(G)$ and proceed to show that (8)–(12) hold. That (9) holds for $\underline{D} = C_{00}(G)$, and thus for $\underline{D} \in \{L_1(G) \cap L_p(G), L_p^{wt}(G), L_p^t(G)\}$ as well, follows from the definition of $L_p^t(G)$. That (8) holds is an elementary consequence of the fact that $C_{00}(G)$ is dense in $L_p(G)$. Since $L_p^{rwt}(G)$ is $||| \ |||_p^{rt}$-closed and $L_p^{lwt}(G)$ is $||| \ |||_p^{lt}$-closed, it is evident that $L_p^{wt}(G)$ is $||| \ |||_p^t$-closed; thus, (10) holds. Since $C_{00}(G) \subset L_p^{wt}(G)$ and since $C_{00}(G) * C_{00}(G) * C_{00}(G)$ is dense in $L_p(G)$, it follows that (11) holds. It remains only to prove (12).

Let $L_1(G)^z$ be the center of $L_1(G)$. It follows via standard arguments that $L_1(G)^z$ consists precisely of those functions $f \in L_1(G)$ such that ${}_xf = f_x$ for all $x \in G$. Since $L_1(G) \cap L_p(G)$ is dense in $L_p(G)$, it is evident that $f * g = g * f$ for all $f \in L_1(G)$ and $g \in L_p(G)$. It was proved in [6] that (for SIN groups) the closed left ideal (= closed right ideal = closed ideal) in $L_1(G)$ generated by $L_1(G)^z$ is just $L_1(G)$.

THEOREM 20. *Let $\{h_\beta\}$ be a minimal approximate identity for $L_1(G)$ lying in $L_p^{wt}(G)$. Then, if C_p is the $\| \ \|_p^t$-unit ball of $L_p^t(G)$ and $g \in L_p^{wt}(G)$,*

$$\lim_{\beta}\sup\left\{\max\{|||h_\beta * f * g - f * g|||_p^t, |||g * f * h_\beta - g * f|||_p^t\}: f \in C_p\right\} = 0.$$

PROOF. Note that, if $g \in L_1(G) \cap L_p(G)$, then $\|g\|_p^t \leq \|g\|_1$.

Choose $l \in L_p^t(G)$ and $h \in L_1(G)$ such that $g = l * h$. Let $\varepsilon > 0$. Choose $\{h_j\}_{j=1}^n \subset L_1(G)^z$ and $\{l_j\}_{j=1}^n \subset L_1(G)$ such that

$$\left\|h - \sum_{j=1}^n l_j * h_j\right\|_1 \leq \varepsilon / \left(2\|l\|_p^t\right).$$

Then

$$\overline{\lim_{\beta}}\ \sup\{|||h_\beta * f * g - f * g|||_p^t : f \in C_p\}$$

$$\leq \overline{\lim_{\beta}}\ \sup\left\{\left|\left|\left|h_\beta * f * l * \sum_{j=1}^n l_j * h_j - f * l * \sum_{j=1}^n l_j * h_j\right|\right|\right|_p^t : f \in C_p\right\} + \varepsilon$$

$$= \overline{\lim_{\beta}}\ \sup\left\{\left|\left|\left|\sum_{j=1}^n (h_\beta * h_j * f * l * l_j - h_j * f * l * l_j)\right|\right|\right|_p^t : f \in C_p\right\} + \varepsilon = \varepsilon.$$

That $\overline{\lim}_\beta \sup\{|||g * f * h_\beta - g * f|||_p^t : f \in C_p\} = 0$ follows analogously. Q.E.D.

It follows that (12) holds. Hence, we know that $QM(L_p^t(G))$ is linearly isometric to $QM(L_p^{wt}(G))$, and that both are topologically isomorphic to $QM(L_p(G);\ L_p^{wt}(G))$. We shall now characterize this latter set more concretely. Let $QM_p(G)$ be the set of all *quasi-multipliers* of type (p, p, p). That is, the set of all $m | L_p(G) \times L_p(G) \to L_p(G)$ separately continuous, bilinear, and such that

(29) $\quad {}_x m(f, g)_y = m({}_x f, g_y)$ for all $f, g \in L_p(G)$ and $x, y \in G$.

THEOREM 21. *We have* $QM_p(G) = QM(L_p(G);\ L_p^t(G)) = QM(L_p(G);\ L_p^{wt}(G)) = QM(L_p(G); L_1(G))$.

PROOF. To facilitate application of Theorem 19, we shall write E, $\underline{B}$, $\underline{A}$, and $\underline{D}$ for $L_p(G)$, $L_p^t(G)$, $L_p^{wt}(G)$, and $L_1(G) \cap L_p^t(G)$ respectively, and $\|\ \|_B$ for $\|\ \|_p^t$. Since $\|f\|_p^t \leq \|f\|_1$ for all $f \in L_1(G) \cap L_p^t(G)$, and since $C_{00}(G) \subset L_1(G) \cap L_p^t(G)$ is $\|\ \|_1$-dense in $L_1(G)$, it is easy to see that $L_1(G) \subset \overline{D}$, the $\|\ \|_B$-closure of $\underline{D}$. Thus, by Theorem 19 and (21), the present Theorem 21 will follow once we have shown that $QM_p(G) = QM(L_p(G);\ L_1(G))$.

Let $F(G)$ be the linear span of the Dirac measures on G. Let $|\ |$ denote the total variation norm on $F(G)$. Clearly, if $m | L_p(G) \times L_p(G) \to L_p(G)$ is separately continuous, then (29) holds if and only if

(30) $m(\mu * f, g * \nu) = \mu * m(f, g) * \nu$ for all $\mu, \nu \in F(G)$.

Let $m \in QM_p(G)$, $f, g \in L_p(G)$ and $h \in L_1(G)$. By (4.11) and (4.12) of [10], together with the analogous right-handed version of (4.2), there is a net $\{\mu_\alpha\}$ in the $|\ |$-ball of radius $\|h\|_1$ of $F(G)$ such that

$$\lim_\alpha \|\mu_\alpha * l - h * l\|_p = 0 = \lim_\alpha \|l * \mu_\alpha - l * h\|_p$$

for all $l \in L_p(G)$. Thus

$$\begin{aligned}\|m(f, g * h) - m(f, g) * h\|_p &= \lim_\alpha \|m(f, g * \mu_\alpha) - m(f, g) * \mu_\alpha\|_p \\ &= \lim_\alpha 0 = 0\end{aligned}$$

and, similarly, $m(h * f, g) = h * m(f, g)$. Thus $m \in QM(L_p(G); L_1(G))$.

Now let $m \in QM(L_p(G); L_1(G))$, $f, g \in L_p(G)$, and $x \in G$. By (4.11) and (4.2), together with the analogous right-handed version of (4.2) in [10], there is a net $\{q_\alpha\}$ in the unit ball of $L_1(G)$ such that

$$\lim_\alpha \|q_\alpha * l - {}_x l\|_p = 0 = \lim_\alpha \|l * q_\alpha - l_x\|_p = 0$$

for all $l \in L_p(G)$. Thus

$$\|m(f, g_x) - m(f, g)_x\|_p = \lim_\alpha \|m(f, g * q_\alpha) - m(f, g) * q_\alpha\|_p = 0$$

and, similarly, $m({}_x f, g) = {}_x m(f, g)$. Thus $m \in QM_p(G)$. Hence $QM_p(G) = QM(E; D)$. Q.E.D.

4. The group algebra of a locally compact group. Let G be a locally compact, Hausdorff, topological group with left Haar measure λ.

THEOREM 22. *The group algebra $L_1(G)$ possesses a minimal ultra-approximate identity.*

PROOF. Let $\{h_\beta\}$ be any minimal approximate identity for $L_1(G)$. Let $T \in LM(L_1(G))$. By Wendel's Theorem, there exists $\mu \in M(G)$ such that

$$T(f) = \mu * f \quad \text{for all } f \in L_1(G).$$

Then, for all $g \in L_1(G)$,

$$\overline{\lim_\alpha} \|g * T(h_\beta) - g * \mu\|_1 = \lim_\alpha \|(g * \mu) - (g * \mu) * h_\beta\|_1 = 0$$

since $L_1(G)$ is an ideal in $M(G)$. Thus, $\{g * T(h_\beta)\}$ is Cauchy in $L_1(G)$. If $S \in RM(L_1(G))$, it is shown analogously that, for each $g \in L_1(G)$, $\{S(h_\beta) * g\}$ is Cauchy in $L_1(G)$. It follows now from Theorem 8 that $\{h_\beta\}$ is an ultra-approximate identity. Q.E.D.

COROLLARY. *The measure algebra $M(G)$ may be identified with $QM(L_1(G))$.*

PROOF. Immediate from Theorems 9 and 23.

5. **C^*-algebras.** To the author's knowledge, [1] and [2] are the only publications dealing directly with quasi-multipliers which will have appeared in print before the present one. Akemann and Pedersen invented the term "quasi-multiplier" in [1] for a C^*-algebra A and showed there that quasi-multipliers may be embedded in the enveloping von Neumann algebra A''. They further distinguished them as precisely those elements of A'' which are continuous on the set of states in A' when the latter bears the relativized weak-* topology.

The paper [2] is concerned with developing a type of duality between the spectrum of a C^*-algebra A and $QM(A)$ through the medium of convergence structures. It was proved there that $QM(A)$ is precisely the set of elements of A'' which are continuous on A' when the latter bears the topology $\sigma\nu(A', A)$, the coarsest topology finer than both the weak-* topology $\sigma(A', A)$ and the topology of convergence *of* norms.

It is interesting that both of these papers arose from the authors' having to correct previous work and that the concept of quasi-multiplier was just what was required to repair the defective theory.

Akemann and Pedersen in [1] state the conjecture that $LM(A) + RM(A) = QM(A)$.

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